$$\left[\begin{array}{ccccc} 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

≥ 1 free var $\rightarrow \infty$ soln
0 free var $\rightarrow$ 1 soln

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & b_1 \\ 0 & 0 & 1 & 4 & 5 & 6 & 7 & b_2 \\ 0 & 0 & 0 & 1 & 5 & 6 & 7 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 & b_4 \end{array}$$

$$\begin{bmatrix} a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

A $m \times n$, $\vec{x}$ $n \times 1$, $\vec{b}$ $m \times 1$

$$A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1}\end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2}\end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn}\end{bmatrix}$$

③ $\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$

2 Definitions of Linear ~~Independence~~ Dependence

Span

- $\text{span}\{v_1, \ldots, v_n\} = \left\{ \sum_{i=1} \alpha_i \vec{v}_i \mid \alpha_i \in \mathbb{R} \right\}$
- set of all linear combos of $\{v_1, \ldots, v_n\}$
- span of a set of vectors is a subspace
- $\text{span}(A) = \text{range}(A)$
- $\text{span}(A) = \text{columnspace}(A)$

Is $\vec{v}$ in the span $\{[\vec{a}], [\vec{b}], [\vec{c}]\}$?

To solve this, aug matrix & GE

$\left[\begin{array}{ccc|c} \vec{a} & \vec{b} & \vec{c} & \vec{v} \end{array}\right] \rightarrow$ if no soln, $\vec{v}$ not in the span

Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ Lin ind?

need to find nullspace. If trivial, LI. If nontrivial, LD

$\left[\begin{array}{ccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{0} \end{array}\right]$

state-transition matrix

$$A = \begin{bmatrix} P_{1\to 1} & P_{2\to 1} & \cdots & P_{n\to 1} \\ P_{1\to 2} & P_{2\to 2} & \cdots & P_{n\to 2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{1\to n} & P_{2\to n} & \cdots & P_{n\to n} \end{bmatrix}$$

$= 1 \rightarrow$ conservative system

$A\vec{x}[n] = \vec{x}[n+1]$ (time n, time $n+1$)

$A^{-1}\vec{x}[n] = \vec{x}[n-1]$ if the inverse exists

↳ note inverse is UNIQUE

Calculating Matrix Inv.

$\left[\begin{array}{c|c} A & I_n \end{array}\right] \rightarrow ge \rightarrow \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$

note: A^{-1} doesn't have to be in RREF

note: for $\underset{m \times n}{A} = \underset{m \times n}{B}\ \underset{n \times m}{C}$, A^{-1} DNE because C has more columns than rows $\Rightarrow$ LD

Basis

- for $\{v_1, \ldots, v_n\} = \mathcal{S}$, the vectors in $\mathcal{S}$ are a basis for $\mathbb{V}$ if ① they're LI ② their span is $\mathbb{V}$
- minimal set of spanning vectors
- for $\mathbb{R}^N$, N LI vectors form a basis

Dimension

- dimension($\mathbb{V}$) equals # vectors in its basis
- $\dim(\mathbb{R}^N) = N$

Subspace

$\mathbb{U}$ is a subspace of $\mathbb{V}$ if: ① contains $\vec{0}$ ② closed under vector + ③ closed under scalar $\times$

subspace basis = LI vectors spanning $\mathbb{U}$

subspace dimension = # vectors in basis

Columnspace

$\text{Col}(A)$ where $\underset{m \times n}{A}$
= span n columns of A
= range(A)

form a subspace for $\mathbb{R}^m$

Rowspace

= span n rows of A

Rank

$= \dim(\text{col}(A))$ = # pivots in RREF
$= \dim(\text{range}(A))$ — can be at most $\min(m,n)$ if A an $n \times m$ matrix
$= \dim(\text{span}(A))$ (# columns)

Rank-Nullity Thm

$\dim(\text{range}(A)) + \dim(N(A)) = n$ (A matrix $m \times n$, row × col)

Nullspace

set of all $\vec{x}$ s.t. $A\vec{x} = \vec{0}$

- if $\vec{x} = \vec{0}$ is only soln, trivial nullspace
- solve for free vars, write as a vector, tada!

ex:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_2, x_4, x_5 free vars

$x_2 = \alpha$, $x_4 = \beta$, $x_5 = \gamma$

$x_1 = -\alpha + 2\beta - 3\gamma$
$x_2 = \alpha$
$x_3 = \beta - \gamma$
$x_4 = \beta$
$x_5 = \gamma$

write as vector sum

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\alpha + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\beta + \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\gamma$$

$N(A) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\dim(N(A)) = 3$

Eigenstuff

- to calculate λs, solve $\det(A - \lambda I) = 0$ for λ
- to calculate $\vec{v}_\lambda$ solve $(A - \lambda I)\vec{v} = \vec{0}$ for $\vec{v}$

$(A - \lambda I)\vec{x} = 0$ $\qquad A\vec{x} = \lambda\vec{x}$

① Find λs – for an $n \times n$ matrix, we should have n λs
② Find eigenvectors corresponding by plugging in $\lambda_1, \ldots, \lambda_n$ into $(A - \lambda I)$
– if a matrix has m eigenvalues, all eigenvectors are LI
– if $\lambda = 0 \rightarrow$ not invertible, nontrivial nullspace
$\lambda = 1 \rightarrow$ steady state
– for a 2×2 matrix: $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$ – basis for $N(A)$ = eigenvectors
– distinct eigenvectors form a subspace – repeated e-values can have 1 or 2 e-vecs

Steady-state $\vec{x}^* = P\vec{x}^*$

– to find steady state, subsitute $\lambda = 1$, solve for nullspace and that's the st-state

ex:

$$A - \lambda I = \begin{bmatrix} -1 & 0 & 0 & 1/2 \\ 1/2 & -1 & 0 & 0 \\ 1/3 & 1/2 & -1 & 1/2 \\ 1/3 & 1/2 & 0 & -1 \end{bmatrix} \quad (\lambda = 1)$$

$(A - \lambda I)\vec{x}^* = 0$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2/3 & 0 \\ 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 = 2x_4 \\ x_2 = 2/3\,x_4 \\ x_3 = 3/2\,x_4 \end{array} \rightarrow \vec{x}^* = \begin{bmatrix} 2 \\ 2/3 \\ 3/2 \end{bmatrix} x_4$$

If you ~~start~~ start pumps with A_0, B_0 and C_0, what's the assoaicated steady-state?

① $A_0 + B_0 + C_0 = D$ ② given $x^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\alpha$, $x_1 + x_2 + x_3 = E$ ③ $D = E\alpha$. Solve for α. ④ Multiply x^* by α to get ss for A_0, B_0 & C_0

Predicting system behavior for ~~[illegible]~~ initial states

$A^n\vec{x} = \alpha(\lambda^n\vec{x})$

- $\lambda > 1$: $\vec{x}[n] \rightarrow \infty$ exponential growth
- $\lambda = 1$: $\vec{x}[n] \rightarrow k\vec{v}$ (steady-state)
- $|\lambda| < 1$: $\vec{x}[n] \rightarrow \vec{0}$ exponential decay
- $\lambda = 1$: $\vec{x}[n] = 0\vec{v} \rightarrow \vec{0}$ instant disappearance

$\vec{x}[0] = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$ $\qquad \vec{x}[n] = \alpha_1\lambda_1^n\vec{v}_1 + \alpha_2\lambda_2^n\vec{v}_2 + \cdots + \alpha_n\lambda_n^n\vec{v}_n$

① given an initial state, solve for $\alpha_1, \ldots, \alpha_n$ $\left[\begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & \vec{s}[0] \end{array}\right]$ ② Plug $\alpha_1, \ldots, \alpha_n$ into $\vec{x}[n]$ eqn and ~~[illegible]~~ lambdas and $\vec{v}_i$

Rotation Matrix

$$A_R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Change of Basis

$\vec{F}^{(v)} = V^{-1}U\vec{F}^{(u)}$ $\qquad \vec{F}^{(v)} = V^{-1}\vec{F}$ $\qquad T = V^{-1}U$ (transformation matrix)

Diagonalization

– matrix $T_{n \times n}$ is diagnolizable iff it has n LI e-vectors with corresponding e-vals

$A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$ ($\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$) $\quad D$ = diagonal matrix of e-vals ($\lambda_1, \lambda_2, \ldots$)

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$$

$T = ADA^{-1}$
$T^k = AD^kA^{-1}$

Procedure
① compute $(\vec{v}, \lambda)$ pairs of T
② make sure e-vecs are LI (can use GE and nullspace to make sure it's trivial)
③ make A and A^{-1} using e-vecs
④ make D out of $\lambda_1, \ldots, \lambda_n$
⑤ multiply and get T^k

Equivalent Statements – LI for $A_{n \times n}$ matrix

- $A\vec{x} = \vec{0}$ only when $\vec{x} = 0$
- $A^2 \neq 0$
- doesn't has an e-val of 0
- A has n pivot positions
- columns/rows form a basis for $\mathbb{R}^n$
- spans $\mathbb{R}^n$
- $\text{rank}(A) = n$
- A invertible
- A has trivial nullspace
- A has LI columns
- A is full rank
- $\det(A) \neq 0$
- $A\vec{x} = \vec{b}$ has a unique soln
- $\text{col}(A) = \mathbb{R}^n$

Thm: $\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \mathbb{R}^2$

Known:
$\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ set of all $\vec{b}$ that can be written as
$\mathcal{S} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\alpha, \beta \in \mathbb{R}$

To show:
$\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \mathbb{R}^2$
any $\vec{b}$ in $\mathbb{R}^2$ can be represented using the span $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Want: All $\vec{b}$ to belong to the set $\mathcal{S}$

$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & -1 & b_2 \end{array}\right]$

$\rightarrow ge \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{b_1 + b_2}{2} \\ 0 & 1 & \frac{b_1 - b_2}{2} \end{array}\right] \rightarrow \alpha = \frac{b_1 + b_2}{2}$ $\quad \beta = \frac{b_1 - b_2}{2} \Rightarrow$ every $\vec{b} \in \mathbb{R}$ can be represented as a linear combo of $\alpha, \beta \Rightarrow \vec{b} \in \mathcal{S}$ ■

Thm: If col(A) are LD, then $A\vec{x} = \vec{b}$ doesn't have a unique soln

Known:
$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$

$\vec{a}_k = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_{k-1}\vec{a}_{k-1} + c_{k+1}\vec{a}_{k+1} + \cdots + c_n\vec{a}_n$

Want:
$A\vec{x} = \vec{b}$ doesn't have unique soln → either 0 or 2+
If possible, let a soln be $\vec{s}$.

rewrite as matrix

$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \rightarrow A\left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}\right) = \vec{0} \rightarrow A\begin{bmatrix} \vdots \end{bmatrix} = \vec{0}$ (k position)

call the vector $\vec{\omega}$: $A\vec{\omega} = 0$

we know $A\vec{s} = \vec{b} \rightarrow A\vec{s} + \vec{0} = \vec{b} \rightarrow A\vec{s} + A\vec{\omega} = \vec{b} \rightarrow A(\vec{s} + \vec{\omega}) = \vec{b}$ $\quad \vec{s} \neq \vec{\omega} \rightarrow$ also a soln!
$\Rightarrow \vec{s}$ is not a unique soln ■

Thm: $A\vec{x} = \vec{b}$ has 2+ solns then columns of A are LD

Known:
$A\vec{x} = \vec{b}$ has 2 distinct solutions $\vec{s}$ & $\vec{\omega}$ but both solns
$A\vec{s} = \vec{b}$, $A\vec{\omega} = \vec{b} \rightarrow A\vec{\omega} = A\vec{s}$
$\rightarrow A(\vec{\omega} - \vec{s}) = \vec{0}$
Let $\vec{\omega} - \vec{s} = \vec{y} \rightarrow \vec{y} \neq \vec{0} \rightarrow \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \rightarrow A\vec{y} = \vec{0}$
$\rightarrow \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \vec{0} \rightarrow y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_n\vec{a}_n = \vec{0}$
At least one $y \neq 0$. call this y_k.
$y_k\vec{a}_k = y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_n\vec{a}_n$
$\vec{a}_k = \frac{y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_n\vec{a}_n}{y_k} \Rightarrow \vec{a}_k$ is lin. combo of other cols ■

Want:
columns of A are LD
$\vec{a}_k = c_1\vec{a}_1 + \cdots + c_n\vec{a}_n$

Thm: If col(A) are LD, A is not invertible

Known: col(A) are LD
$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = 0$

Want: A^{-1} DNE
Assume A^{-1} Does exist

Let A^{-1} exist. $\rightarrow A^{-1}A = AA^{-1} = I \rightarrow [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$
$\rightarrow A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \rightarrow A^{-1}A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A^{-1}\vec{0} \rightarrow I\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$
contradiction! → A^{-1} can't exist ■

Thm: If A is invertible, unique soln exists

Known
$A^{-1}A = AA^{-1} = I$

To show:
① at least 1 soln
② only other soln (equivalent)

Guess: $\vec{s} = A^{-1}\vec{b}$ is a soln
Check: $A\vec{s} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b} \Rightarrow \vec{s}$ is a soln
Say $\vec{\omega} \neq \vec{s}$ is also a soln
$A\vec{\omega} = \vec{b} \rightarrow A^{-1}A\vec{\omega} = A^{-1}\vec{b} \Rightarrow \vec{\omega} = A^{-1}\vec{b}$, $\vec{\omega} = \vec{s}$

Tips for Proofs

- unique soln? proof by contradiction
- want columns of A? matrix mult
- if $A\vec{x} = \vec{0}$ then you can add this to pretty much anything

If Matrix A is invertible, its col are LI

A^{-1} exists $\rightarrow A\vec{x} = \vec{0}$ only when $\vec{x} = \vec{0}$
say $A\vec{v} = \vec{0} \rightarrow A^{-1}A\vec{v} = A^{-1}\vec{0} \rightarrow \vec{v} = \vec{0}$ ■

form a basis

① $\vec{v}_1, \vec{v}_2$ are LI ② $\vec{v}_1, \vec{v}_2$ span all of $\mathbb{R}^2$

If possible, let $\vec{v}_1$ & $\vec{v}_2$ be LD

$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 = \vec{0}$ → say $\alpha_1 \neq 0$ → $\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}\vec{v}_2$

Multiply by A.

$A\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}A\vec{v}_2$

$A_1\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}\lambda_2\vec{v}_2$

$\lambda_1\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}\lambda_2\vec{v}_2$

→ plug in $\vec{v}_1$ into the eqn.

$-\frac{\alpha_2}{\alpha_1}\lambda_1\vec{v}_2 = -\frac{\alpha_2}{\alpha_1}\lambda_2\vec{v}_2$

$\lambda_1 = \lambda_2$

↳ contradiction!

→ therefore $\vec{v}_1, \vec{v}_2$ are LI

To show they span all of $\mathbb{R}^2$:

$[\vec{v}_1 \ \vec{v}_2 \mid \vec{x}]$ → $V = [\vec{v}_1 \ \vec{v}_2]$

→ V is an invertible matrix

⇒ $[V \mid \vec{x}]$ has a unique soln

→ therefore $\vec{x} \in \text{span}\{\vec{v}_1, \vec{v}_2\}$

⇒ $\{\vec{v}_1, \vec{v}_2\}$ form a basis for $\mathbb{R}^2$

If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ are LD vectors in in $\mathbb{R}^n$, then $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n\}$ are LD.

Known:

$\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$

Show:

$A\vec{v}_k = \sum_{\ell \neq k} \beta_\ell \vec{v}_\ell$

$A\vec{v}_i = A\left(\sum_{j \neq i} \alpha_j \vec{v}_j\right)$

$= \sum_{j \neq i} A(\alpha_j \vec{v}_j)$

$A\vec{v}_i = \sum_{j \neq i} \alpha_j (A\vec{v}_j)$ ⇒ linear combo exists

QED

If $\vec{v}_1, \vec{v}_2$ solns to $A\vec{x} = \vec{b}$, $\vec{v}_1 + \vec{v}_2$, $\vec{b}$ must be $\vec{0}$.

Known: $A\vec{v}_1 = \vec{b}$, $A\vec{v}_2 = \vec{b}$

Want: $\vec{b} = \vec{0}$

$A(\vec{v}_1 + \vec{v}_2) = \vec{b}$ → $A\vec{v}_1 + A\vec{v}_2 = \vec{b}$

⇒ $\vec{b} + \vec{b} = \vec{b}$ ⇒ $\vec{b} = \vec{0}$

QED

$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$

say $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

$\vec{q} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$

$= \alpha_1(\vec{v}_1 + \vec{v}_2) + (-\alpha_1 + \alpha_2)\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$

⇒ $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$

say $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$

$\vec{r} = \beta_1(\vec{v}_1 + \vec{v}_2) + \beta_2(\vec{v}_2) + \cdots + \beta_n\vec{v}_n$

$= \beta_1\vec{v}_1 + (\beta_1 + \beta_2)\vec{v}_2 + \cdots + \beta_n\vec{v}_n$

$\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

If A invertible, unique A^{-1}

Known: $AA^{-1} = A^{-1}A = I$

Want: A^{-1} unique

say B_1, B_2 inverses of A, $B_1 \neq B_2$

$AB_1 = B_1A = I$ $AB_2 = B_2A = I$

$B_2AB_1 = B_2B_1A$

$(B_2A)B_1 = B_2(B_1A)$

$B_1 = B_2$

→ contradiction, A^{-1} must be unique

If $QP = I$ and $RQ = I$, then $P = R$.

$QP = RQ$

$RQP = RRQ$

$(RQ)P = R(RQ)$

$I \quad P = R \quad I$

If a system of K reservoirs has columns that sum to one, then if s is the total amount of water at timestep n, then the total amount of water is S at timestep n+1

Known

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$a_{11} + a_{21} = 1$

$a_{12} + a_{22} = 1$

$x_1[n] + x_2[n] = s$

$\vec{x}[n+1] = A\vec{x}[n]$

Consider product $A\vec{x}[n] = \vec{x}[n+1]$

Want

$x_1[n+1] + x_2[n+2] = s$

mat mul → $\begin{bmatrix} a_{11}x_1[n] + a_{12}x_2[n] \\ a_{21}x_1[n] + a_{22}x_2[n] \end{bmatrix}$

Transpose

- eigenvalues remain the same across transposes

Applying Matrices

- go from right to left. ex.

$ABCD\vec{x}$

$(A(B(C(D\vec{x}))))$

④ ③ ② ①

Given unknown Matrix A, given $A\vec{u}_1 = A\vec{u}_2 = \vec{p}$, find $\vec{u}$ s.t. $A\vec{u} = \vec{0}$ where $\vec{u} \neq \vec{0}$.

$A\vec{u}_1 - A\vec{u}_2 = \vec{0}$

$A(\underbrace{\vec{u}_1 - \vec{u}_2}_{\vec{u}}) = \vec{0}$

$\vec{u} = \vec{u}_1 - \vec{u}_2$

Matrix inverse Properties

- $AA^{-1} = A^{-1}A = I$
- $(A^{-1})^{-1} = A$
- $(KA)^{-1} = K^{-1}A^{-1}$ $K \in \mathbb{R}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

More steady-state:

$\vec{s}[0] = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$

To decompose $\vec{s}[0]$ into the eqn (given $\vec{s}[0]$):

$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \vec{s}[0]$

↳ do GE

- $A\vec{s}[0]$ has lambdas ($A\vec{s}[0] = \lambda\vec{s}[0]$)

$\lambda_i\vec{v}_i = A\vec{v}_i$

⇒ $A^{-1}\lambda_i\vec{v}_i = \vec{v}_i$

$A^{-1}\vec{v}_i = \frac{1}{\lambda_i}\vec{v}_i$

~~Is $\vec{v}$ in the column space of A when a = 3?~~

Is $v = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$ in $C(A)$ when a = 3?

$A = \begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix}$ a = 3 $v = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are LI & span $\mathbb{R}^2$ → yes

4 b

Solve for smallest possible identical e-vals for A.

$A = \begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix}$ → $\det(A - \lambda I) = 0$ → $\lambda^2 - (2+a)\lambda + (2a+1) = 0$

→ $\lambda = \frac{2 + a \pm \sqrt{(2a+a)^2 - 4(2a+1)}}{2}$ → since we want identical evals, everything under sqrt = 0.

→ $(2a + a)^2 - 4(a2 + 1) = 0$ → solve for a → a = 0, 4

→ want a minimizing e-vals → plug in a to the quadratic eqn formula w/ the λs → ($a = 4$ → $\lambda = 3$) & ($a = 0$ → $\lambda = 1$) → $\boxed{a = 0}$

Find all vals for x s.t. A has a trivial nullspace

$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 6 \\ 0 & 1 & x \end{bmatrix}$ → ge → $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 2x \end{bmatrix}$ → want LI so $x \neq 0$

Pluto — 7c

Given a transformation, what is the transformation matrix that created the transform?

ex: $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ → $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

and

$\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ → $\begin{bmatrix} -3 \\ -1.5 \end{bmatrix}$

rewrite:

$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -3 \\ -1.5 \end{bmatrix}$

→ do mat mul, solve for a, b, c & d and plug into A

$a = \frac{2}{3}$ $b = 0$ $c = 0$ $d = \frac{2}{3}$

⇒ $A = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix}$